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A formal solvability of a coupling equation for PDEs of Briot-Bouquet type

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Abstract

We study couplings for a pair of a partial differential equation of Briot-Bouquet type in the t variable and its model equation, without assuming the analytic dependency in t . In this report, we concentrate on the formal solvability —the existence of a formal solution of a special form— of a coupling equation on one side indicated as (Ψ) . The precise statement concerning the convergence, together with a similar question on the reversed equation, that is, the coupling equation on the other side indicated as (Φ) , will be published elsewhere.

§ 1. Introduction

The notion of coupling equations was introduced by the third author [2], for a theory of a class of transformations between nonlinear partial differential equations of normal form in complex domains. It was extended in [3] and [4] for partial differential equations of Briot-Bouquet type.

In the original coupling theory, the analytic dependency in the independent variables of the original equations plays an important role, and the solutions to a coupling equation were treated as formal power series of a special form in infinitely many variables.

Recently, using a functional analytic approach with the notion of infinite dimensional holomorphy, we studied the coupling equations for partial differential equations of normal form in the t variable, without the requirement of the analytic dependency in

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t . (See [1]). As for couplings for partial differential equations of Briot-Bouquet type in the t variable without analytic dependency in t , we have not succeeded to introduce a similar functional analytic approach for the solvability result. However, as for coupling equations for an equation and its model equation, we succeed to solve them under a weaker assumption of the dependency in t .

In this report, we focus to illustrate the formal solvability of a coupling equation (Ψ) .

§ 2. Coupling equations for PDEs of Briot-Bouquet type

Let us briefly recall the solvability results in [3], of coupling equations for PDEs of Briot-Bouquet type in a complex domain.

A partial differential equation of an unknown $u(t, x)$

$$(F) \quad t \frac{\partial u}{\partial t} = F(t, x, u, \frac{\partial u}{\partial x})$$

with a given differentiable function $F(t, x, z_0, z_1)$ of four variables in a neighborhood of the origin is said to be of *Briot-Bouquet type* in the t variable, if F satisfies the so-called *Briot-Bouquet condition*

$$(BB) \quad F(0, x, 0, 0) = 0, \quad \frac{\partial F}{\partial z_1}(0, x, 0, 0) = 0.$$

In this case, the *characteristic exponent* of (F) is defined by

$$(CE) \quad \lambda(x) := \frac{\partial F}{\partial z_0}(0, x, 0, 0),$$

and F is written as

$$(2.1) \quad F(t, x, z_0, z_1) = \sum_{k \geq 1} F_k(t, x, z_0, z_1) = a(x)t + \lambda(x)z_0 + F_{\geq 2}(t, x, z_0, z_1).$$

Here F_k denotes the homogeneous part of degree k in the Taylor expansion of F in (t, z_0, z_1) variables, and $F_{\geq 2} = \sum_{k \geq 2} F_k$.

Among such equations sharing the same characteristic exponent $\lambda(x)$, a simple example is

$$(M) \quad t \frac{\partial v}{\partial t} = \lambda(x)v,$$

which is actually a linear ordinary differential equation in t with a parameter x . We call (M) a model equation of (F).

In [3], third author considered the case that F is a holomorphic function in (t, x, z_0, z_1) in a neighborhood of the origin in \mathbb{C}^4 , and studied couplings between (F) and (M).

Actually, he considered the correspondences

$$\begin{aligned}\Phi : u &\mapsto v, & v(t, x) &= \Phi[u](t, x) := \phi(t, x, ((\frac{\partial}{\partial x})^i u(t, x))_{i \in \mathbb{N}}), \\ \Psi : v &\mapsto u, & u(t, x) &= \Psi[v](t, x) := \psi(t, x, ((\frac{\partial}{\partial x})^i v(t, x))_{i \in \mathbb{N}}),\end{aligned}$$

defined via $\phi(t, x, z)$ and $\psi(t, x, z)$ with $z = (z_i)_{i \in \mathbb{N}} = (z_0, z_1, \dots)$, which are regarded as “holomorphic functions of infinitely many variables”, and studied the condition for Φ to transform solutions of (F) into those to (M), and that for Ψ to transform solutions vice versa. Such conditions were described as *coupling equations*: $\phi(t, x, z)$ and $\psi(t, x, z)$ should formally satisfy

$$\begin{aligned}(\Phi) \quad & t \frac{\partial \phi}{\partial t} + \sum_{m \in \mathbb{N}} D^m F(t, x, z_0, \dots, z_{m+1}) \cdot \frac{\partial \phi}{\partial z_m} = \lambda(x) \phi, \\ (\Psi) \quad & t \frac{\partial \psi}{\partial t} + \sum_{m \in \mathbb{N}} D^m (\lambda(x) z_0) \cdot \frac{\partial \psi}{\partial z_m} = F(t, x, \psi, D\psi),\end{aligned}$$

where D denotes a formal vector field of infinitely many variables, defined by

$$D = \frac{\partial}{\partial x} + \sum_{i \in \mathbb{N}} z_{i+1} \frac{\partial}{\partial z_i}.$$

A notion of “holomorphic functions of infinitely many variables” for ϕ and ψ in this situation was interpreted as a formal power series involving infinitely many variables (t, z) of form

$$(2.2) \quad \phi, \psi \in \sum_{k \geq 1} \mathcal{O}_x(\mathbb{D}_R)[t, z_0, \dots, z_{k-1}]_k,$$

where $\mathcal{O}_x(\mathbb{D}_R)$ denotes the space of holomorphic functions on $\mathbb{D}_R := \{x \in \mathbb{C} \mid |x| < R\}$, and $\mathcal{O}_x(\mathbb{D}_R)[t, z_0, \dots, z_{k-1}]_k$ denotes the space of homogeneous polynomial of degree k in the (t, z_0, \dots, z_{k-1}) variables with coefficients in $\mathcal{O}_x(\mathbb{D}_R)$. In other words, ψ and ϕ admit decompositions into homogeneous parts and into monomials in (t, z) of form

$$(2.3) \quad \psi(t, x, z) = \sum_{k \geq 1} \psi_k(t, x, z_0, \dots, z_{k-1}) = \sum_{k \geq 1} \sum_{\substack{(i, j) \in \mathbb{N} \times \mathbb{N}^k, \\ i + |j| = k}} \psi_{i, j}(x) t^i z_0^{j_0} \cdots z_{k-1}^{j_{k-1}},$$

For example, the homogeneous part of degree 1 of ψ reads

$$\psi_1(t, x, z_0) = \psi_{1,0}(x)t + \psi_{0,1}(x)z_0.$$

By substituting these decompositions, we can reduce the coupling equations (Φ) and (Ψ) into recursive relations in $k \in \mathbb{N}$. In fact, for example, the coupling equation (Ψ)

for F of form (2.1) reads

$$\begin{aligned} & \left\{ t \frac{\partial}{\partial t} + \lambda(x) \left(\sum_{m \in \mathbb{N}} z_m \frac{\partial}{\partial z_m} - 1 \right) + \sum_{m \in \mathbb{N}} \sum_{p=0}^{m-1} \lambda_{m,p}(x) z_p \frac{\partial}{\partial z_m} \right\} \psi \\ & = a(x)t + F_{\geq 2}(t, x, \psi, D\psi), \end{aligned}$$

where $\lambda_{m,p}(x) := \binom{m}{p} \left(\frac{d}{dx} \right)^{m-p} \lambda(x)$, and the corresponding recursive relation for $\psi_k = \sum_{i+|j|=k} \psi_{i,j}(x) t^i z^j$ is

$$\begin{aligned} & \sum_{i+|j|=k} \left\{ i + \lambda(x)(|j| - 1) \right\} \psi_{i,j}(x) t^i z^j \\ & + \sum_{i+|j|=k} \sum_{m \in \mathbb{N}} \sum_{p=0}^{m-1} j_m \lambda_{m,p}(x) \psi_{i,j}(x) t^i (z_p / z_m) z^j \\ & = \text{terms determined by } F \text{ and } \{\psi_\ell\}_{1 \leq \ell < k}. \end{aligned}$$

We can interpret the left hand side as a linear operator with diagonal 1st term and off-diagonal 2nd term, applied to a family of unknowns $\psi_{i,j}$ for $(i, j) \in \mathbb{N} \times \mathbb{N}^k$, $i + |j| = k$. Therefore, if $i + \lambda(x)(j - 1)$ never vanishes for any $(i, j) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0), (0, 1)\}$ and for any x under consideration, the coupling equation (Ψ) admits a formal power series solution of form (2.3) satisfying $\psi_{0,1}(x) = \beta(x)$ for any holomorphic function $\beta(x)$.

In [3], the coupling equation (Φ) was also studied and similar formal power series solutions were constructed. Moreover, under so-called the *Poincaré condition* on the characteristic exponent $\lambda(x)$:

$$\begin{aligned} (\text{P}) \quad & \exists \sigma > 0, \exists R > 0, \forall x \in \overline{\mathbb{D}}_R, \forall (i, j) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0), (0, 1)\}, \\ & |i + \lambda(x)(j - 1)| \geq \sigma(i + j), \end{aligned}$$

the convergence results were proved for such formal solutions ϕ and ψ , and some applications were obtained.

§ 3. A case with non-analytic dependency in t

We study the case that F is not necessarily analytic in t . Let $F(t, x, z_0, z_1)$ be a continuous function defined in a neighborhood of the origin in $\mathbb{R}_t \times \mathbb{C}_{(x, z_0, z_1)}^3$, which is holomorphic in x, z_0 and z_1 . If moreover F satisfies the Briot-Bouquet condition (BB), the equation (F) is similarly said to be a PDE of Briot-Bouquet type in the t variable with the characteristic exponent $\lambda(x)$ defined by (CE). Note that we shall further pose a differentiability assumption on F in the t variable in order to study the solvability of coupling equations. However, for introducing the notion of couplings, it suffices to assume the continuity in t .

Between the equation (F) and the model equation (M) sharing the same characteristic exponent $\lambda(x)$, the notion of the coupling can be introduced, and we get the same coupling equations (Φ) and (Ψ) , completely in the same manner as in the case of complex analytic dependency in Section 2.

On the other hand, we can not expand F into the Taylor series in the (t, z_0, z_1) variables like (2.1). Moreover, we can neither expect our functions ϕ and ψ to be a formal power series in the (t, z) variables like (2.2) and (2.3).

Now we pose the assumption that $F \in C_t^{m+1} \mathcal{O}_{(x, z_0, z_1)}$ for a positive integer m , and that

$$0 < \operatorname{Re} \lambda(0) < m, \quad \lambda(0) \notin \mathbb{Z}.$$

Note that the differentiability assumption for F can be relaxed to $F \in C_t^m \mathcal{O}_{(x, z_0, z_1)}$ by introducing the notion of “continuous solution” to the equation (Ψ) , while we skip it here. Moreover, in this report, we restrict ourselves to the case $m = 1$, for the sake of simplicity. That is, we assume that F is C^2 in t , and that the characteristic exponent $\lambda(x)$ satisfies $0 < \operatorname{Re} \lambda(0) < 1$. In this case, F can be written as

$$(3.1) \quad F(t, x, z_0, z_1) = a(t, x)t + \lambda(x)z_0 + \sum_{k \geq 2} F_k(t, x, z_0, z_1),$$

$$(3.2) \quad F_k(t, x, z_0, z_1) = \sum_{i+j_0+j_1=k} F_{i,j_0,j_1}(t, x) t^i z_0^{j_0} z_1^{j_1},$$

instead of (2.1), with $a(t, x), F_{i,j_0,j_1}(t, x) \in C_t^1 \mathcal{O}_x$. Note that it is possible to take the sum only for $i = 0, 1$ in (3.2), and that the expansion (3.1) is not unique. As for an example of the non-uniqueness, a function tz_0^2 can belong to F_3 as a monomial $t^1 z_0^2 z_1^0$ with a coefficient 1, or alternatively to F_2 as a monomial $t^0 z_0^2 z_1^0$ with a coefficient t .

Note also that the hypothesis $0 < \operatorname{Re} \lambda(0) < 1$ implies

$$(RP_1) \quad \exists \sigma > 0, \exists R > 0, \forall x \in \overline{\mathbb{D}}_R, \forall (i, j) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0), (0, 1)\}, \\ \operatorname{Re}\{i + \lambda(x)(j - 1)\} \geq \sigma(i + j),$$

which is a stronger condition than (P).

In this situation, we want to find a solution ψ to (Ψ) , of form

$$(3.3) \quad \psi(t, x, z) = \sum_{k \geq 1} \sum_{\substack{(i,j) \in \mathbb{N} \times \mathbb{N}^k, \\ i+|j|=k}} \psi_{i,j}(t, x) t^i z_0^{j_0} \cdots z_{k-1}^{j_{k-1}},$$

with $\psi_{i,j}(t, x) \in C_t^1 \mathcal{O}_x$ instead of (2.3).

Actually, we study the following equation:

$$(\hat{\Psi}) \quad t_d \frac{\partial \hat{\psi}}{\partial t_d} + t_s \frac{\partial \hat{\psi}}{\partial t_s} + \sum_{m \in \mathbb{N}} D^m(\lambda(x)z_0) \cdot \frac{\partial \hat{\psi}}{\partial z_m} = \hat{F}(t_d, t_s, x, \hat{\psi}, D\hat{\psi}),$$

with a given function

$$(3.4) \quad \hat{F}(t_d, t_s, x, z_0, z_1) = a(t_d, x)t_s + \lambda(x)z_0 + \sum_{k \geq 2} \hat{F}_k(t_d, t_s, x, z_0, z_1),$$

$$\hat{F}_k(t_d, t_s, x, z_0, z_1) = \sum_{i+j_0+j_1=k} F_{i,j_0,j_1}(t_d, x) t_s^i z_0^{j_0} z_1^{j_1}.$$

and an unknown

$$(3.5) \quad \hat{\psi}(t_d, t_s, x, z) = \sum_{k \geq 1} \hat{\psi}_k(t_d, t_s, x, z_0, \dots, z_{k-1})$$

$$= \sum_{k \geq 1} \sum_{\substack{(i,j) \in \mathbb{N} \times \mathbb{N}^k, \\ i+|j|=k}} \psi_{i,j}(t_d, x) t_s^i z_0^{j_0} \cdots z_{k-1}^{j_{k-1}},$$

where F_{i,j_0,j_1} and $\psi_{i,j}$ are given in (3.1) and in (3.3), respectively. For a solution $\hat{\psi}$ to the equation $(\hat{\Psi})$, we can show that $\psi(t, x, z) := \hat{\psi}(t, t, x, z)$ solves the equation (Ψ) .

Theorem 3.1. *Let $\beta(x)$ be a germ of a holomorphic function in x in a neighborhood of $x = 0$. Then, there exists a unique formal solution $\hat{\psi}$ of form (3.5) to the equation $(\hat{\Psi})$, satisfying $\psi_{0,1}(t_d, x) = \beta(x)$.*

Remark. For any formal solution $\hat{\psi}$ of form (3.5) to the equation $(\hat{\Psi})$, $\psi_{0,1}(t_d, x)$ is necessarily independent of t_d .

Let us give the idea of the proof.

The equation $(\hat{\Psi})$ reads, for the homogeneous part of degree 1,

$$\left(t_d \frac{\partial}{\partial t_d} + 1 - \lambda(x)\right) \psi_{1,0}(t_d, x) t_s + t_d \frac{\partial}{\partial t_d} \psi_{0,1}(t_d, x) z_0 = a(t_d, x) t_s,$$

or equivalently

$$\left(t_d \frac{\partial}{\partial t_d} + 1 - \lambda(x)\right) \psi_{1,0}(t_d, x) = a(t_d, x), \quad t_d \frac{\partial}{\partial t_d} \psi_{0,1}(t_d, x) = 0,$$

and for the higher degree parts,

$$\sum_{i+|j|=k} \left\{ t_d \frac{\partial}{\partial t_d} + i + \lambda(x)(|j| - 1) \right\} \psi_{i,j}(t_d, x) t^i z^j$$

$$+ \sum_{i+|j|=k} \sum_{m \in \mathbb{N}} \sum_{p=0}^{m-1} j_m \lambda_{m,p}(x) \psi_{i,j}(t_d, x) t^i (z_p / z_m) z^j$$

$$= \text{terms determined by } \hat{F} \text{ and } \{\hat{\psi}_\ell\}_{1 \leq \ell < k}.$$

Therefore, we have $\psi_{0,1}(t_d, x) = \beta(x)$ for an arbitrary holomorphic function $\beta(x)$, and the other $\psi_{i,j}$ are determined uniquely in a recursion according to the order for (i, j) determined by $i + |j|$ and $\sum_p (p+1)j_p$, since $(t_d \frac{\partial}{\partial t_d} + \mu(x))$ with $\operatorname{Re} \mu(x) > 0$ admits an inverse

$$f(t_d, x) \mapsto \int_0^1 q^{\mu(x)-1} f(qt_d, x) dq,$$

for $C_t^1 \mathcal{O}_x$ -functions.

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